Fatigue Life Prediction

22.1 Introduction

Vibration loads will cause oscillating stresses and forces internally in the spacecraft structure. Oscillating stresses can lead to fatigue problems. Mathematical models are available, such as the linear cumulative damage rule of Palgren-Miner (published by Palgren in 1924 and by M.A. Miner in 1945 [Juvinall 1967]) in combination with the fatigue (s-N) curve of a design material, to predict the so-called fatigue life of a structural part under a oscillating environment [Bishop 2000]. The linear cumulative damage rule assumes that the total life of a structural part can be predicted by merely adding up the percentage of life consumed by each stress cycle.

In this chapter the linear cumulative damage rule based on the stress-life approach is discussed. s-N fatigue analyses recapitulated are based on:

- Time domain stress-life fatigue life estimation
- Frequency domain model
- Random vibration narrow band solution

22.2 Palmgren-Miner Linear Cumulative Damage Rule

There are of course numerous failure modes. One of them is failure of the structure due to material fatigue. So the question remains as to what is fatigue actually? Fatigue is an occurrence where structures crack or even break (prematurely) due to alternating loads. Alternating micro plastic deformations that damage the material structure locally causes fatigue. As the alternating loads continue to increase, the deformations can accumulate to become micro cracks or fractures. The Palmgren-Miner rule (life-fraction rule) is applied to predict the lifetime of a structure or a structural part that is affected by linear cumulative damage. At a certain stress level s_i, making use of the so-called s-N (Wöhler or fatigue) curve, one can predict the permissible number of cycles N_i. Additionally, one can calculate the number of real "cycles" that take place for a certain stress s_i during launch or the test. According to the cumulative damage model, the following must hold during the lifetime of a spacecraft structure (to prevent the structure from failing due to fatigue):

$$D = \sum_i \frac{n_i}{N_i} \tag{22.1}$$

where D is the cumulative damage at i load cases with n_i oscillations or cycles at the stress level s_i and N_i the allowable number of oscillations at the stress level s_i. Miner's rule states that $D = 1$ at fatigue failure. Miner cites numerous tests with factors at $0.7 \le \sum_i \frac{n_i}{N_i} \le 1.2$ [Juvinall 1967].

The total number of cycles which cause fatigue failure is N_T and is defined by

$$N_T = \sum_i n_i . \tag{22.2}$$

Equation (22.1) can be rewritten with $D = 1$ as follows, [Richards 1968]

$$N_T \sum_i \frac{n_i}{N_T N_i} = 1 . \tag{22.3}$$

But the probability of occurrence of stress s_i is defined by

$$p_i = \frac{n_i}{N_T} . \tag{22.4}$$

Hence

$$N_T \sum_i \frac{p_i}{N_i} = 1 \tag{22.5}$$

The number of allowable cycles N_i at the stress level s_i can be obtained from the s-N curve, in general, described by the following equation

$$N(s)s^b = c . \tag{22.6}$$

A load event causes the same damage regardless of where it occurs in the overall load history.

22.3 Analysis of Load-time Histories

The time responses (time history) are in general rather irregular. Cycling counting is to be done for example by the rain flow method [Bishop 2000]. The rain flow or range pair-range counting method [Jonge 1982, AGARD 1983] is widely used to decompose the irregular time history into equivalent sets of block loading. The number of cycles in each block are recorded in a stress range histogram. The stress range histogram can then be used in the Palmgren-Miner calculations. We try to explain the procedure with the help of two examples [www.me.iastate.edu].

Example

Given a block of transient loading in Fig. 22.1 and

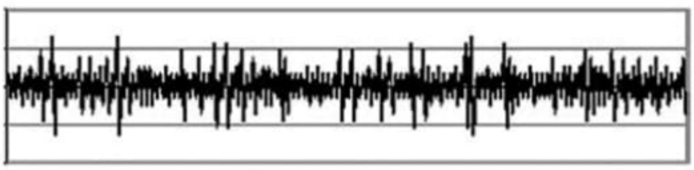

Fig. 22.1 Transient loading

assuming it is reduced in a constant amplitude events, i.e. with cycle counting, see Fig. 22.2.

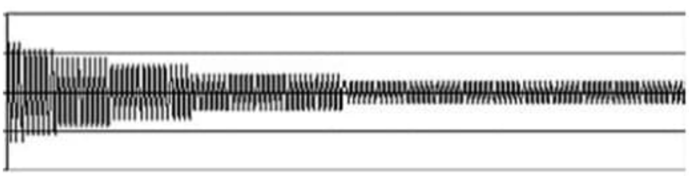

Fig. 22.2 Constant amplitude events

It is possible to extract a block of stress amplitudes associated with a number of cycles. This is shown in Table 22.1.

Table 22.1 Block of stress amplitudes and number of cycles

Block	
Stress amplitudes (Mpa)	**Number of cycles**
448	3
379	6
310	10
241	15
172	28
103	63

End of example

Example

A steel component has an endurance limit (10^6 cycles) at 207 Mpa and a 10^3 cycles strength of 510 Mpa. The s-N ($Ns^b = c$) curve is defined and based on the former information. If the known values are substituted in the expression for the s-N curve we obtain two equations

$$b\log(207\times10^6) - \log c = -\log(10^6)$$

and

$$b\log(510\times10^6) - \log c = -\log(10^3)$$

It is found that $c = 5.1011x10^{69}$ and $b = 7.6609$.

Determine the life of the component (in number of blocks) and how much of the overall damage is contributed by each of the stress levels. Use the s-N method, linear damage rule and Miner's constant $D = 1$.

The block is given in Table 22.2.

Table 22.2 Block of stress amplitudes and cycles

Block	
Stress amplitudes (Mpa)	**Number of cycles**
482	3
400	8
310	50
269	350
221	1000

Table 22.3 Fatigue life prediction per block

Stress Amplitude s (MPa)	number of cycles n_i	$N_i \;\; Ns^b = c$	$D_i = \frac{n_i}{N_i}$	Percent total Damage %
482	3	1540	0.0019	22.8
400	8	6430	0.0012	14.6
310	50	45330	0.0011	12.9
269	350	134380	0.0026	30.5
221	1000	605710	0.0017	19.3
	Total Damage per Block $\sum_{i=1}^{5} D_i$		0.0085	100.0
	Blocks to failure $\sum_{i=1}^{5} D_i = 1$		116.9	

The cumulative damage per block associated with the Palgren-Miner rule block is calculated in Table 22.3. The predicted life is about 117 blocks.

22.4 Failure due to Sinusoidal Vibrations

The enforced acceleration at the base of the spacecraft, solar arrays, instruments, equipment, etc. will result in responses in the frequency domain.

This response is the steady state solution of the be equations of motion. The stress and or force responses in the structure are illustrated in Fig. 22.3.

The frequency responses in an elastic structure with damping is written as

$$\sigma(\omega) = H_{\sigma\ddot{u}}(\omega)\ddot{u}(\omega) \text{ and } F(\omega) = H_{F\ddot{u}}(\omega)\ddot{u}(\omega) \tag{22.7}$$

The frequency response function (FRF) $H_{\sigma\ddot{u}}(\omega)$ and $H_{F\ddot{u}}(\omega)$ can be evaluated with a general purpose finite element programme assuming a unitary base excitation $\ddot{u}(\omega) = 1$. In general, the sinusoidal base excitation is in a frequency range between 5–100 Hz.

On a shaker, the sinusoidal enforced acceleration is applied to a structure with a certain (logaritmic) sweep rate with the unit octave/min, see Fig. 22.4.

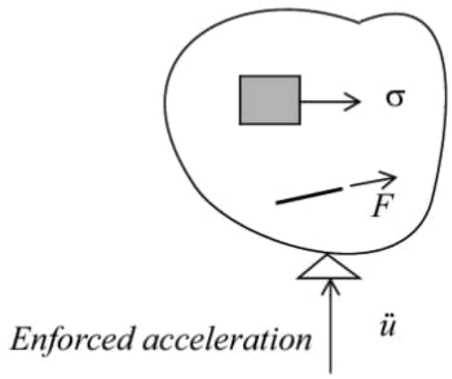

Fig. 22.3 Enforced acceleration

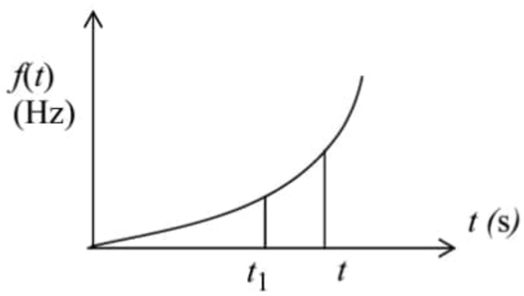

Fig. 22.4 Sweep rate

The logarithmic ratio of the frequency ratio is equal to a constant times the time difference

$$\ln\left\{\frac{f(t)}{f_1}\right\} = K(t - t_1)\,. \tag{22.8}$$

From (22.8) we derive that

$$f(t)dt = \frac{df}{K}. \tag{22.9}$$

The sweep rate is defined as the number of octaves per minute, i.e. n Oct/min. If there are n Oct/min, (22.8) can be written $\ln(2^n) = K60$. Thus the constant K is

$$K = \frac{n\ln(2)}{60} = 0.0116n\,. \tag{22.10}$$

The number of cycles in a certain time domain can be obtained by

$$N(t) = \int_{t_{\text{ref}}}^{t} f(t)dt = \frac{1}{K}\int_{f_{\text{ref}}}^{f} df = \frac{f - f_{\text{ref}}}{K} = \frac{\Delta f}{K}. \qquad (22.11)$$

The number of cycles per Hertz is constant all over the frequency range and is given by

$$N(\Delta f{=}1) = \frac{1}{K} = \frac{86.6}{n}.$$

If the frequency is swept from the frequency f_1 to f_2 the average stress becomes

$$\sigma_{\text{average}}(\Delta f_i) = \frac{1}{2}\{\sigma(f_i) + \sigma(f_{i-1})\}. \qquad (22.12)$$

The number of allowable cycles at a stress level σ_{average} can be obtained from the fatigue curve $N(s)s^b = c$

$$N_{\text{s-N}}(\Delta f_i) = \frac{c}{\sigma^b_{\text{average}}(\Delta f_i)}. \qquad (22.13)$$

The cumulative damage due to sinusoidal stresses, swept with a certain sweep rate *n,* can be calculated with the aid of the following expression

$$D_{\text{sinusoidal}} = \sum_i \frac{N(\Delta f_i)}{N_{\text{s-N}}(\Delta f_i)}. \qquad (22.14)$$

The fatigue life associated with the cumulative damage $D_{\text{sinusoidal}}$ can calculated by

$$T_{\text{sinusoidal}} = \frac{1}{K}\ln\left(\frac{f_{\max}}{f_{\min}}\right). \qquad (22.15)$$

22.5 Failure due to Narrow-banded Random Vibrations

The expected damage due to the number of peaks, while the stress *s* lies between a certain stress level s and $s + ds$ during a time T, is given by:

$$D(s) = \frac{n(s)}{N(s)}, \qquad (22.16)$$

where $n(s)$ is the number of cycles occurring at stress level s = a and the number of allowable cycles $N(s)$ is taken from the s-N curve (22.6)

$$N(s) = \frac{c}{s^b} . \tag{22.17}$$

The number of cycles occurring at stress level $s_a = a$ is given by

$$n(s_a) = \nu_a^+ T , \tag{22.18}$$

where

$$n(s) = \nu_s^+ - \nu_{s+ds}^+ = \frac{-d\nu_s^+}{ds} ds = \nu_0^+ \frac{s}{\sigma_s^2} e^{-\frac{s^2}{2\sigma_s^2}} ds . \tag{22.19}$$

The number of positive crossings at stress level $s_a = a$, for a narrow banded process, is given by

$$\nu_a^+ = \nu_0^+ e^{-\frac{s_a^2}{2\sigma_s^2}} , \tag{22.20}$$

with the number of positive zero crossings ν_0^+, with the power spectral density function $W_s(f)$ given by

$$\nu_0^+ = \left(\frac{\int_0^\infty f^2 W_s(f) df}{\int_0^\infty W_s(f) df} \right)^{\frac{1}{2}} . \tag{22.21}$$

The linear cumulative damage due to $n(s_a)$ can be calculated by

$$D(s) = \frac{n(s)}{N(s)} = \frac{\nu_s^+ T}{N(s)} = \frac{s \nu_0^+ T e^{-\frac{s^2}{2\sigma_s^2}}}{N(s)\sigma_s^2} ds \tag{22.22}$$

where σ_s^2 is the variance of the stress s.

According to Palmgren-Miner, the total expected cumulative damage amounts to:

$$E[D(T)] = \frac{\nu_0^+ T}{\sigma_s^2} \int_0^\infty \frac{s e^{-\frac{s^2}{2\sigma_s^2}}}{N(s)} ds . \tag{22.23}$$

In combination with (22.6), the s-N curve, (22.23) representing the expected value of the cumulative damage becomes

$$\bar{D}(T) = \frac{\nu_0^+ T}{c\sigma_s^2}\int_0^\infty s^{b+1} e^{-\frac{s^2}{2\sigma_s^2}} ds = \frac{\nu_0^+ T}{c}\{\sqrt{2}\sigma_s\}^b \Gamma\left(1+\frac{b}{2}\right) \tag{22.24}$$

with the Gamma function[1] $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ and σ_s is the rms value of the induced (to be investigated) stress component.

To find the fatigue damage due to a wide-banded process the damage based on the narrow-banded asumption can be estimated first, and correction is then made correspondingly [Jiao 1990].

Steinberg proposed in [Steinberg 1978] an approximation

$$D = \nu_0^+ T\left[\frac{0.683}{N_{1\sigma}} + \frac{0.271}{N_{2\sigma}} + \frac{0.043}{N_{3\sigma}}\right], \tag{22.25}$$

where $N_{1\sigma}$ is the number of allowable oscillations at a 1σ stress level, etc.

Assuming that available cumulative damage is $\bar{D}(T) \approx 1$ when the structure fails due to fatigue, then the expected life time $\bar{T}$ can be calculated.

$$\bar{T} = \frac{c}{\nu_0^+\{\sqrt{2}\sigma_s\}^b \Gamma\left(1+\frac{b}{2}\right)}. \tag{22.26}$$

With

$$h_1 = \frac{\nu_0^+}{c}\{\sqrt{2}\sigma_s\}^b \Gamma\left(1+\frac{b}{2}\right) \text{ and } h_2 = \frac{1}{c}\{\sqrt{2}\sigma_s\}^b \Gamma\left(1+\frac{b}{2}\right)\sqrt{\frac{\nu_0^+\psi_1(b)}{\zeta}},$$

we are able to calculate the standard deviation of the fatigue life distribution [Sun 1996]

$$\sigma_T = \frac{h_2\sqrt{h_2^2 + 4h_1}}{6h_1^2}, \tag{22.27}$$

with $\zeta = \frac{1}{2Q}$ the modal damping ratio, and $\psi_1(b)$ a function of b, tabulated in Table 22.4.

---

1. $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(1) = 1$, $\Gamma(m+1) = m!$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Table 22.4 $\psi_1(b)$ [Crandall 1963]

b	**1**	**3**	**5**	**7**
$\psi_1(b)$	0.0414	0.3690	1.2800	3.7200
b	9	11	13	15
$\psi_1(b)$	10.700	31.5000	96.7000	308.0000

Assume a Gaussian probability density function (pdf) of the fatigue life *T*, then

$$f_N = \frac{1}{\sigma_T\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{T-\bar{T}}{\sigma_T}\right)^2} . \qquad (22.28)$$

The normal distribution can be transferred to the standard normal pdf, using the transformation $z = \frac{T-\bar{T}}{\sigma_T}$, [Bain 1987], hence

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} , \qquad (22.29)$$

and the cumulative density function (CDF) becomes

$$\Phi(z) = \int_{-\infty}^{z} \phi(t)dt , \qquad (22.30)$$

with [Stange 1970]

$$\Phi(-z) = 1 - \Phi(z) . \qquad (22.31)$$

The reliability with respect to fatigue life becomes

$$R_N = \int_T^{\infty} f_N(t)dt = 1 - \Phi\left(\frac{T-\bar{T}}{\sigma_T}\right) = 1 - \Phi(z) . \qquad (22.32)$$

Example

A mass-spring system, with a mass *m* and a spring stiffness *k* of the rod, is random excited at the base with $W_{\ddot{u}}(f)$. The amplification factor is $Q = 10$. A natural frequency of $f_o = 50$Hz is selected and the spring stiffness of the spring is $k = \frac{EA}{L} = 1\times10^{-4}$ N/m, with a surface area of $A = 10^{-4}$ m^2. The mass *m* is

$$m = \frac{k}{(2\pi f_o)^2} = 101.3 \text{ kg}.$$

The random base excitation is $W_{\ddot{u}}(f) = 10\ (\text{m/s}^2)^2/\text{Hz}.$, in a frequency domain from 5–2000Hz.

With the aid of Miles' equation the acceleration of the mass m can be calculated:

$$\ddot{x}_{\text{rms}} = \sqrt{\frac{\pi}{2} f_o Q W_{\ddot{u}}(f_o)} = \sqrt{\frac{\pi}{2} x50x10x10} = 88.6\ \text{m/s}^2$$

The rms value of the stress $s_{\text{rms}} = \sigma_s$ in the rod then becomes:

$$s_{\text{rms}} = \sigma_s = \frac{m\ddot{x}_{\text{rms}}}{A} = \frac{101.3x88.6}{1\times10^{-4}} = 8.98x10^7\ \text{Pa}$$

For a simple mass-spring system $\nu_0^+ = f_o = 50$ Hz.

The expected lifetime T can be calculated as follows: The S-N curve is given and is $N(s)s^b = c$, with $b = 4$, $c = 1.56\ 10^{37}$, $T = \dfrac{c}{\nu_0^+\{\sqrt{2}\sigma_s\}^b\Gamma\left(1+\frac{b}{2}\right)} = 600$ s.

With $b = 4$ the function $\psi_1(4) = \dfrac{\psi_1(3) + \psi_1(5)}{2} = 0.64$

The standard deviation of the predicted expected fatigue life is

$$\sigma_T = \frac{h_2\sqrt{h_2^2 + 4h_1}}{6h_1^2} = 12.4\ \text{s}$$

Assume $z = \dfrac{T - \bar{T}}{\sigma_T} = \dfrac{\bar{T} - 3\sigma_T - \bar{T}}{\sigma_T} = -3$, thus with $\Phi(-z) = 1 - \Phi(z)$, $\Phi(-3) = 1 - \Phi(3) = 1 - 0.9987 = 0.0013$.

Thus the reliability figure for a fatigue life $\bar{T} - 3\sigma_T$ becomes $R_N = 1 - \Phi(-3) = 0.9987$.

The total number of cycles n then becomes $n = \nu_0^+ T = 50 * 600 = 30000$ after which the rod will collapse due to fatigue.

For a sinusoidal stress alternation with constant amplitude, constant frequency and a total of 30000 cycles, then it follows from the s-N curve that the permissible stress amplitude is equal to: $s = \left(\dfrac{c}{N}\right)^{\frac{1}{b}} = \left(\dfrac{1.56x10^{37}}{30000}\right)^{\frac{1}{4}} = 1.51x10^8$ Pa.

End of example

Example Steinberg approach

This example is taken from [Steinberg 1978]. A mounting bracket that is to hold a transformer of 3.5 kg.

The bracket and box are illustrated in Fig. 22.5. The total length of the bracket is $2L = 0.4$ m, the width $b = 0.1$ m and the thickness $t = 0.01$ m. The bracket has been manufactured from Al-alloy with a Young's modulus $E = 70$ GPa. The mass of the box is $M = 3.5$ kg. The power spectral density of the enforced acceleration is constant in the frequency range between 50-200 Hz and is given by $W_{\ddot{u}} = 0.1$ g^2/Hz. The assembly will be exposed to an enforced vibration test of duration $T = 10$ hrs. The amplication facor or transmissibility is $Q = 25$. The s-N curve of the construction material steel is given by $Ns^{1.585} = 1.4231x10^{19}$

Determine whether the bracket design is satisfactory for the specified vibration level and duration of the test, and also how long it can withstand such a test.

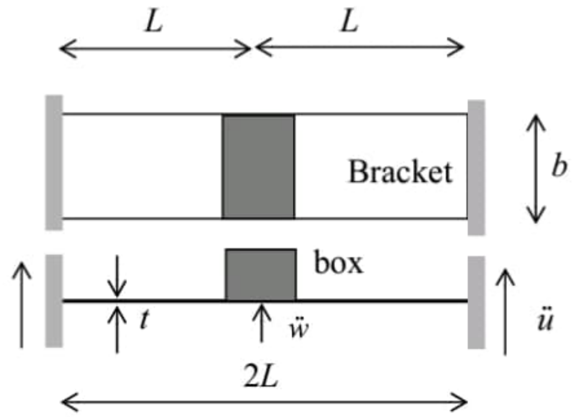

Fig. 22.5 Box mounted on bracket

Initially the natural frequency of the bracket is calculated while ignoring the mass of the bracket. The second moment of area I of the bracket is

$$I = \frac{1}{12}bt^3 = 8.333x10^{-9} \text{ m}^4.$$

The deflection of the clamped-clamped beam in the mid of the beam under a gravitation load Mg is given by

$$\delta = \frac{MgL^3}{24EI},$$

where $g = 9.81$ m/s^2.

The natural frequency can be calculated using

$$f_o = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{1}{2\pi}\sqrt{\frac{24EI}{ML^3}} = 112.5 \text{ Hz}.$$

The M-line of a beam clamped at both sides with a force in the middle of the beam is illustrated in Fig. 22.6. The maximum absolute value of the bending moment is $M_{max} = \frac{FL}{4}$. The maximum corresponding bending stress at the extreme fibre distance $e = \frac{t}{2}$ is given by

$$\sigma_{bending,max} = \frac{M_{max}e}{I} = \frac{M_{max}t}{2I}.$$

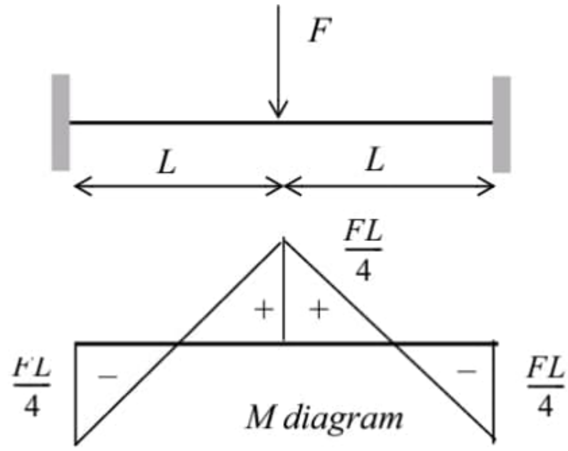

Fig. 22.6 M-line

It is now required to calculate the inertia force F. This rms value of the inertia force is given by

$$F_{rms} = M\ddot{w}_{rms}$$

where $\ddot{w}_{rms}$ is the rms acceleration of the box. The rms acceleration can be calculated using the Miles' equation

$$\ddot{w}_{rms} = \sqrt{\frac{\pi}{2}f_o Q W_{\ddot{u}}(f_o)} = \sqrt{\frac{\pi}{2}x112.5x25x0.1} = 21.0\,\mathrm{g}$$

The rms value of the inertia force F_{rms} becomes

$$F_{rms} = M\ddot{w}_{rms} = 3.5x21.0x9.81 = 721.8\,\mathrm{N}.$$

If a stress concentration factor $K = 2.2$ is taken into account, the maximum bending stress is

$$\sigma_{bending,rms} = \frac{KM_{max}t}{2I} = \frac{KFLt}{8I} = \frac{2.2x721.8x0.2x0.010}{8.333\times10^{-9}} = 4.764\times10^{7}\,\mathrm{Pa}.$$

The number of allowable cyles at stress levels can be obtained from the s-N curve $Ns^{1.585} = 1.4231x10^{19}$:

- $1\sigma = \sigma_{\text{bending,rms}}$ is $N_{1\sigma} = 9.631 \times 10^6$
- $2\sigma = 2\sigma_{\text{bending,rms}}$ is $N_{2\sigma} = 3.21 \times 10^6$
- $3\sigma = 3\sigma_{\text{bending,rms}}$ is $N_{3\sigma} = 1.688 \times 10^6$

Finally it is possible to calculate the linear cumulative damage in accordance to the Palgren-Miner rule and applying (22.25)

$$D = f_0 T\left[\frac{0.683}{N_{1\sigma}} + \frac{0.271}{N_{2\sigma}} + \frac{0.043}{N_{3\sigma}}\right] = 0.733$$

How many hours the vibration test can last until the cumulative damage $D = 1$

$$T_{D=1} = \frac{T}{D} = \frac{10}{0.733} = 13.65 \text{ hrs.}$$

End of example